The Mass of a Spin Vortex in a Bose-Einstein Condensate

Ari M. Turner[1]
[1]*Department of Physics, University of California, Berkeley, California 94720, USA*
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In contrast to charge vortices in a superfluid, spin vortices in a ferromagnetic condensate move inertially (if the condensate has zero magnetization along an axis). The mass of spin vortices depends on the spin-dependent interactions, and can be measured as a part of experiments on how spin vortices orbit one another. For Rb^{87} in a 1 μm thick trap, $m_v \sim 10^{-21}$kg.

Vortices, with their long-life and concentration of energy, often provoke comparison to particles. But does their *motion* fit the analogy? Ordinary vortices in a fluid or superfluid do not move inertially, as particles do, because their motion is Magnus-force dominated. For example, in the absence of a force, they do not move. When a force is applied, they move perpendicular to it, a situation which is described by first order differential equations[1]. In fact, the motion of a pair of vortices is a miniature version of how Descartes[2] explained the motion of planets, with the sun causing the ether to whirl around, dragging the planets at the same speed. On the other hand, spinor superfluids[3, 5**?**] made out of laser-cooled atoms can have spin-current vortices. These vortices will be argued to obey Newton's laws at low speeds; in particular they have a mass, which determines their resistance to being accelerated.

A mass can also be defined for a vortex in a superconductor or a superfluid and it may play a role in determining the oscillation frequency of vortex lattices and the tunneling rate of vortices[6]. Observing the mass for such vortices is much more subtle than for spin vortices, though, because the vortex inertia is nearly overcome by the Magnus force. The picture of a moving spin vortex described here is inspired partly by the motion of vortex rings[7] and Ref. [8]'s study of that motion based on the Gross-Pitaevskii equations.

We will focus on the case of spin 1 ferromagnetic atoms in a two-dimensional condensate. A magnetic field is applied along z to stabilize the spin vortices, via the quadratic Zeeman effect. (We will take the condensate to be in the xy plane for definiteness, but the spin and spatial coordinates can be chosen independently.) The ground states of these atoms have the form

$$\psi = e^{i\theta_C + i\theta_S S_z} \begin{pmatrix} \sqrt{n_1} \\ \sqrt{n_0} \\ \sqrt{n_{-1}} \end{pmatrix} = \begin{pmatrix} e^{i(\theta_C+\theta_S)}\sqrt{n_1} \\ e^{i\theta_C}\sqrt{n_0} \\ e^{i(\theta_C-\theta_S)}\sqrt{n_{-1}} \end{pmatrix}. \quad (1)$$

The values n_1, n_0, n_{-1} correspond to the optimal proportions of the three spin states (the total density is $n = n_1 + n_0 + n_{-1}$), and are fixed. If the magnetization along the z-axis, $M_z = n_1 - n_{-1}$, vanishes, then the spin is in the xy-plane, as is preferred by the quadratic Zeeman effect[5]. The overall phase of the wave function (which is not observable) is θ_C, and $-\theta_S$ is the azimuthal angle of the spin. (To see this, calculate $< S_x >$ and $< S_y >$ for this state.) The latter angle can be measured by scattering polarized light off the condensate.

There are two types of vortices in a spinor condensate in a magnetic field: a charge vortex, described by $\theta_C = \pm\phi, \theta_S = 0$ and a spin vortex described by $\theta_C = 0, \theta_S = \pm\phi$ (and observed in a rubidium-87 condensate[9]). Here, ϕ is the azimuthal angle centered on the vortex core. Fig. 1 illustrates these vortices:

$$\psi_C = \begin{pmatrix} e^{i\phi}\sqrt{n_1} \\ e^{i\phi}\sqrt{n_0} \\ e^{i\phi}\sqrt{n_{-1}} \end{pmatrix} \qquad \psi_S = \begin{pmatrix} e^{i\phi}\sqrt{n_1} \\ \sqrt{n_0} \\ e^{-i\phi}\sqrt{n_{-1}} \end{pmatrix}. \quad (2)$$

The spin texture is uniform around a charge vortex (except near the core); the spin direction rotates by 360° clockwise or counterclockwise in a spin vortex. See Fig. 1.

A spinor condensate is somewhat similar to a mixture of several species of atoms. From this perspective, a spin vortex is a bound state of two opposite vortices in two different components and a charge vortex is a bound state of three vortices.

The velocity fields in the components of a spinor condensate are related:

$$\boldsymbol{u}_1 = \boldsymbol{u}_C + \boldsymbol{u}_S; \quad \boldsymbol{u}_1 = \boldsymbol{u}_C; \quad \boldsymbol{u}_{-1} = \boldsymbol{u}_C - \boldsymbol{u}_S; \quad (3)$$

The flow in the middle component (for spin 1) is the mean of the flows in the other two components. As a simple example, the atoms near a charge vortex flow with velocity $\boldsymbol{u}_m = \frac{\hbar}{M}\nabla\theta_m$ [7]. They all move in the same direction (see Fig. 1a), meaning that there is a net transport of mass. On the other hand, the atoms near a spin vortex do not carry any mass if the condensate has zero magnetization $M_z = n_1 - n_{-1}$, because the atoms of spin ±1 move at equal and opposite speeds. However, there is a net transport of spin, since the amount of angular momentum carried counterclockwise depends on the current of $m = 1$ atoms moving counterclockwise and of $m = -1$ atoms moving clockwise. Vortices can be classified by the amounts θ_C and θ_S wind by, Q_C and Q_S respectively. (In general, the charge and spin currents are given by $J_S = n\boldsymbol{u}_C + M_z\boldsymbol{u}_S$ and $J_S = q_z\boldsymbol{u}_S + M_z\boldsymbol{u}_C$ where $q_z = \psi^\dagger S_z^2\psi$. If $M_z = 0$, charge vortices have only charge currents and spin vortices have just spin currents.)

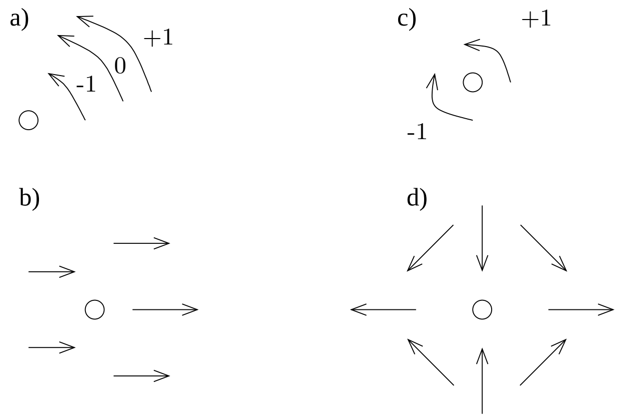

FIG. 1: Charge and Spin Vortices of strength 2π. a), c) The currents of the spin states around charge and spin vortices. b), d) The spin textures around charge and spin vortices.

Now the force on a vortex moving at speed $\boldsymbol{v}$ and tossed about by a flow of charge and spin is given by $\boldsymbol{F} = -\hbar\hat{\boldsymbol{z}} \times [\sum n_m Q_m(\boldsymbol{u}_m - \boldsymbol{v})]$. The term for a given value of m describes the lift force on the vortex in that component: according to Bernoulli's principle higher velocities correspond to lower pressures, so the vortex moves to the side where its velocity field is pointing in the same direction as the relative velocity of the fluid. For a spin vortex of charge Q_S the total force is

$$\dot{\boldsymbol{p}} = \boldsymbol{F}_S - Q_S \boldsymbol{v} \times \hbar M_z \hat{\boldsymbol{z}}, \tag{4}$$

where $\boldsymbol{F}_S$ is a force produced by the background spin current and the second term is the Magnus force, or lift, responding to the vortex's own motion. For a charge vortex of charge Q_C, the force (assuming $M_z = 0$) is

$$\dot{\boldsymbol{p}} = \boldsymbol{F}_C - Q_C \boldsymbol{v} \times \hbar n \hat{\boldsymbol{z}}, \tag{5}$$

where $\boldsymbol{F}_C$ is produced by charge current.

Idiosyncracies of Vortex Motion: Let us compare the motion of charge and spin vortices. A charge vortex in stationary fluid cannot drift along a straight line, because the lift force would push it sideways. Furthermore, the equations of motion take the form of first order differential equations for charge vortices when the inertial term $\dot{\boldsymbol{p}}$ in Eq. (4) is assumed small:

$$\frac{d\mathbf{r}}{dt} = \frac{1}{n\hbar Q_C}\hat{\boldsymbol{z}} \times \boldsymbol{F}_C; \tag{6}$$

motion is perpendicular to applied forces. The vortex velocity required by Eq. (6) turns out to equal the background flow speed, in accordance with Descartes's conception of planetary motion. *The motion of charge vortices is determined once their initial positions are given.*

In contrast, a spin vortex behaves in a Newtonian way, as long as the condensate has zero magnetization. In the absence of spin current, Eq. (4) implies "Newton's first law of spin vortices": *a spin vortex in a charge current can move at any constant speed.* There is no lift to push the vortex off course in an unmagnetized condensate because the component vortices ψ rotate in opposite directions (see Fig. 1c). Now if there is a spin current the lift-forces on the component vortices from the counterpropagating flows *add* to produce a nonzero $\boldsymbol{F}_S$. The solution to $\dot{\boldsymbol{p}} = 0$, $\boldsymbol{v} = \frac{\boldsymbol{F}_S}{\hbar Q_S M_z}$, does not make sense if $M_z = 0$. Therefore the inertial term cannot be neglected and "Newton's second law of spin vortices" results: a *spin-vortex in a spin current must accelerate*, at a rate proportional to $\boldsymbol{F}_S$. The spin force has an electrostatic form:

$$\boldsymbol{F}_{S12} = \frac{\hbar^2 q_z}{M}\frac{Q_{S1}Q_{S2}\hat{\boldsymbol{r}}_{12}}{r_{12}}. \tag{7}$$

One can introduce the vortex mass m_v by assuming that

$$\boldsymbol{p} = m_v \dot{\boldsymbol{r}} \tag{8}$$

at least at low speeds. Hence, the equation of motion prescribes the *acceleration*:

$$m_v \frac{d^2\boldsymbol{r}}{dt^2} = \boldsymbol{F}_S \tag{9}$$

These Newton's laws describe spin vortices in condensates of any spin, as long as $M_z = 0$[19].

Phase Space: The phase space of N vortices seems likely to be $2N$ dimensional, given that the basic equation for the evolution of a spin-1 condensate is the following first-order differential equation:

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2\nabla^2}{2M}\psi + \frac{\partial\mathcal{V}}{\partial\psi^\dagger} \tag{10}$$

Here $\mathcal{V}$ is the potential energy of the atoms. Only the *spatial* coordinates of the vortices seem essential for parameterizing the superfluid wave function.

For charge vortices, the phase space *is* $2N$ dimensional. The motion is even described by Hamilton's equations[10], with the y coordinate of each vortex acting as the momentum conjugate to x! (There are surprising consequence for the thermodynamics of vortices[11, 12].) The precise conjugacy relation reads

$$\{x, y\} = \pm\frac{1}{nh}, \tag{11}$$

where the sign depends on the direction of circulation.

However, Newton's second law implies that N *spin* vortices have a $4N$ dimensional phase space. For ordinary objects, the momenta exist only in an abstract space: a photograph of a ball in the air does not reveal its destination. The future of a spin vortex will also depend on which way it is moving at a given time, so Eq. (10) implies that something about the wave function which is *observable in a single photograph* can be used to *deduce* the vortex's velocity.

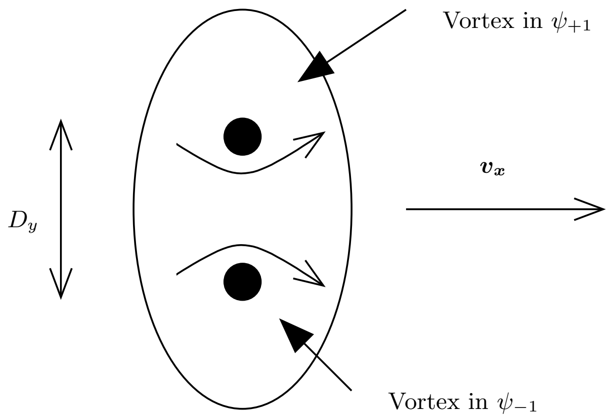

FIG. 2: Seeing the momentum of a spin vortex. The phases of $\psi_{\pm 1}$ increase in the directions of the arrows. The component vortices are pulled apart by the lift force to a distance $D_y \propto p_x$. The stretching is limited by the phase-locking produced by the spin-dependent interaction.

To guess the tell-tale trait, note that the lift forces on the component vortices of a moving spin vortex pull them in opposite directions, but they are restrained from drifting apart completely (see below). Thus, the stretching of the spin vortex increases with its speed (see Fig. 2). The momentum $\boldsymbol{p}$ can be discerned from a snapshot if it is proportional to the stretching:

$$\boldsymbol{p} = -hn_1 \hat{\boldsymbol{z}} \times \boldsymbol{D} \tag{12}$$

where $\boldsymbol{D}$ points between the component vortices. This relation is consistent with the canonical mechanics of the component vortices[13]. According to Eq. (11) $\{x_\pm, y_\pm\} = \pm\frac{1}{n_1 h}$ where $(x_\pm, y_\pm)$ is the location of the component vortex with circulation $\pm 2\pi$. Now the center of mass coordinates x and y of the spin vortices have the ordinary Poisson bracket of zero, but x is conjugate to $p_x = n_1 h(y_+ - y_-)$:

$$\{x, p_x\} = 1. \tag{13}$$

Eq. (12) is correct for a dipole of vortices in a single-component superfluid also, but the velocity of the dipole increases as its momentum (the distance between the vortices) decreases! This unusual fact can be traced back to the logarithmic interaction energy $E \propto \ln D$. Spin vortices behave more normally because the confinement energy is quadratic, so $v = \frac{dE}{dp}$ increases linearly with p.

Eq. (12) holds when the phase-only approximation applies[14]. The general formula for the momentum is $\boldsymbol{p} = -\hat{\boldsymbol{z}} \times \iint d^2\boldsymbol{u}\, \boldsymbol{r}$ curl $\boldsymbol{J}$, where $\boldsymbol{J}$ is the mass current (see [15]). In the phase-only approximation, curl $\boldsymbol{J}$ is a dipole of delta-functions, leading to Eq. (12).

Confinement and the Vortex Mass: Spinor condensates can have special symmetries that ordinary mixtures (e.g., of two different atoms) cannot have[16]. But the higher symmetry is not the only special thing about spinor condensates. Unlike in a mixture, the atomic states can turn into one another in a spinor condensate, as long as the angular momentum does not change: $1 + -1 \leftrightarrow 0 + 0$. This "coherent spin-flipping" process is contained in the potential energy operator, $\mathcal{V} = \frac{1}{2}\alpha_{m_1 m_2} : |\psi_{m_1}|^2 |\psi_{m_2}|^2 : +2\beta\Re(\psi_1^\dagger \psi_{-1}^\dagger \psi_0^2) + q\psi^\dagger S_z^2 \psi - \mu|\psi|^2$.

Coherent spin-flipping locks the phases of the spinor components together[17, 18] and this keeps the two parts of a spin vortex bound together. A nonzero phase $\lambda = (\theta_1 + \theta_{-1} - 2\theta_0)$ costs energy

$$\mathcal{E}_{flip} = 2\beta n_0 \sqrt{n_1 n_{-1}} \cos\lambda, \tag{14}$$

where $\beta < 0$ is the spin-dependent interaction parameter. Since the energy cannot depend on a phase unless particle numbers are not conserved, it makes sense that this comes from the spin-flipping reactions.

This "coherent chemistry" interpretation of Eq. (14) is complemented by an argument which views the atoms as classical magnets: the magnetization in the xy-plane depends on the relative phases between the spin components and decreases as λ increases. This goes against the ferromagnetic propensities of the atoms.

Now the *potential energy* of the stretched vortex

$$E \sim |\beta| n^2 D^2 \tag{15}$$

can be reinterpreted as kinetic energy on account of Eq. (12). At low speeds, $E = \frac{p^2}{2m_v}$, so

$$m_v \sim \frac{\hbar^2}{\beta} \sim M\frac{w}{\Delta a}. \tag{16}$$

The second expression is obtained by relating β to the width of the condensate w and the scattering-length difference Δa [18]. In this case, m_v is of the same same order as the mass of the atoms in the vortex core, $nl_m^2 M$, where l_m is the magnetic healing length. For rubidium in a $1\mu m$ wide trap with $\Delta a \sim 1$Å, $m_v \sim 10^{-21}$ kg.

Measuring the Mass: Now consider the consequences of spin vortices' inertial motion. Spin vortices of opposite signs orbit around one another whereas opposite charge vortices push each other along parallel lines[1]. As for planets, the orbits have different shapes depending on the initial momenta of the vortices. Specifically, when the two vortices move on a circle, attracting each other according to a $\frac{1}{r}$ force law, the period is proportional to its radius. Balancing the attraction Eq. (7) against the centrifugal force $\dot{p} = p\omega$, gives

$$vp(v) = \frac{2\pi n_1 \hbar^2}{M} \tag{17}$$

where v is the speed of both vortices. This remains true when $p(v)$ is nonlinear. The left hand side is an increasing function of v, so $v = v_{circ}$ is determined. This speed is independent of the orbit radius.

Measuring v_{circ} will give an *estimate* for the vortex mass appearing in Eq. (8). However, the linear relation for $p(v)$ is not reliable at the speed v_{circ} because it is on the order of the speed of spin-waves in the condensate.

A more accurate way to measure the mass of spin vortices is to observe their motion when the magnetization is not zero, but is small. Then the lift force due to the vortex's motion looks like the Lorentz force from a small magnetic field, $\boldsymbol{B_A} = -M_z\hbar\hat{\boldsymbol{z}}$. A single vortex will therefore follow cyclotron orbits with the period

$$\tau = \frac{m_v}{\hbar M_z}, \tag{18}$$

If the magnetization is 5% and the other parameters of the condensate are those given above, this period comes out to be .3 sec.

Limiting Velocity: A final idea for an experiment is to study the motion of a rapidly moving vortex. Such a measurement allows hypothetical inhabitants of the superfluid to measure the velocity of their "ether." Spin vortices can move inertially only up to a certain velocity relative to the condensate.

Imagine pushing a spin vortex, starting from rest. After a certain amount of acceleration, the vortex may become unstable, so that all the additional energy goes into producing spin waves. Alternatively, the vortex may remain stable, and absorb all the energy. The energy goes into stretching the components apart until the vortex becomes needle-shaped. The velocity is bounded in this case as well. E is proportional to the vortex-length $D \propto p$ and a linear dispersion implies a finite velocity.

Spin-wave dissipation seems to be the fate of a vortex in a condensate where the atomic interactions in the Hamiltonian are rotationally symmetric (of form $\frac{1}{2}\alpha(\psi^\dagger\psi)^2 + \frac{1}{2}\beta(\psi^\dagger\mathbf{S}\psi)^2$). This conclusion is based mainly on numerical solutions of the Gross-Pitaevskii equations for steadily moving vortices with $\beta = -.3\alpha, q = .5\mu$. (see Fig. 3). The computer did not find solutions past $v_c = .65\sqrt{\frac{\mu}{M}}$, which is close to the spin wave speed.

Experimental conditions can be adjusted so that the vortex does not radiate sound. The symmetry of the condensate has to be broken further, by displacing the traps for the three S_z states into parallel planes. They still have to overlap some to allow for interspecies conversion. This trap set-up maintains the distribution of the atoms among the spin states more rigidly[20]. The component vortices can now separate arbitrarily far without any instability.

Displaced clouds also allow one to use the component-vortex picture more rigorously[14]. When $|\beta| \ll \alpha_{min}$ (α_{min} is the smallest eigenvalue of $\alpha_{m_1m_2}$), the phase-only approximation $|\psi_m| = const.$ applies. One can then derive the $p(D)$ relation Eq. (12) and show that the speed of an infinitely stretched vortex is $\frac{4n_0}{\pi}\sqrt{\frac{|\beta|}{nM}}$.

To summarize, each charge-vortex has only two spatial degrees of freedom because the lift force overcomes

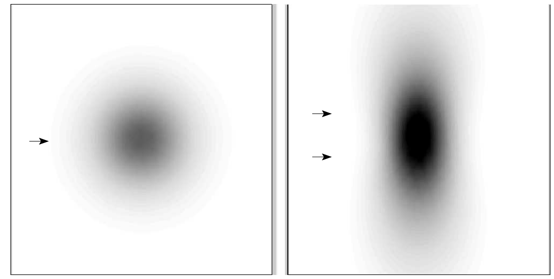

FIG. 3: Computer solution for vortex cores moving at speeds $v = 0, .45\sqrt{\frac{\mu}{M}}$, with rotationally symmetric interactions. The darker regions have the higher energy densities. The arrows indicate where ψ_1 and ψ_{-1} vanish. Note that the energy density extends past these zeros, where the phase mismatch ends. This energy must come from varying magnetization, which eventually causes spin wave emission.

the inertia. In contrast spin vortices in an unmagnetized condensate behave like classical particles because they are made up of oppositely rotating vortices whose total Magnus force cancels. The internal stretching between these components give rise to the mass. Spin vortices have four degrees of freedom: the center of mass coordinates and the momenta which are proportional to the distortion of the vortex core. The vortices behave like classical particles at low speeds, but betray their composite origin when accelerated sufficiently.

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[19] More precisely, the cross-stiffness K_{CS} must equal zero. When the Gross-Pitaevskii equations apply, $K_{CS} \propto M_z$. Also, $K_{CS} = 0$ for any time-reversal symmetric state.
[20] Separating the different spin species increases the energy cost for changing the proportions of atoms in the states from $q \sim \beta n$ (the spin-dependent interaction energy) to αn. With the clouds together, the atoms can all move into the $+1$ state without changing the α term. But the density of the atoms increases when they all go into the same spin state if the traps are displaced.